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Estimating Objects from Shape Relation Measures

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Summary

We measure objects in relation to reference objects by parameterizing measures such as intersection count, area and inner and outer contour length by the distance r from the reference objects to the objects of interest. This expands the notion of these geometric measures giving a more fine-grained interpretation of the shape of the objects and solves issues such as when the domain is bounded by an observation window. One interesting question that arises from this technique is the inverse problem. Is it possible to take the measures and recreate the object, if so, is the object unique and if not, what is the family of shapes under which the measures are invariant? We here tackle the initial step towards a better understanding of the problem.

The Forward Problem

Let X denote the objects of interest and Y the reference objects. We then define the r parameterization by the family Y^r of objects (observed domain) by

$$Y^r = \{\alpha \in \mathbb{R}^d \mid \inf_{y \in Y} d(\alpha, y) \leq r\} \ . \tag{1}$$

Here $d(\alpha, y)$ denotes the distance between α and y, typically the Euclidean distance. The points in X having at most distance r from Y is thus the intersection $X \cap Y^r$. We measure $X \cap Y^r$ by a measure $\mu(X, Y^r)$. In all our applications, μ has the form

$$\mu_{\varepsilon,\varepsilon'}(X, Y^r) = \mathcal{H}^{d-\varepsilon-\varepsilon'}(\partial^\varepsilon X \cap \partial^{\varepsilon'} Y^r) \ , \tag{2}$$

where $\mathcal{H}^k$ denotes the n-dimensional Hausdorff measure, $\varepsilon, \varepsilon' \in \{0, 1\}$. The interpretation of $\mathcal{H}^k$ and $\mu_{\varepsilon,\varepsilon'}(X, Y^r)$ in 2D is shown in the below table.

$\partial^\varepsilon X \cap \partial^{\varepsilon'} Y^r$	Interpretation of $\mu_{\varepsilon,\varepsilon'}(X, Y^r)$
$X \cap Y^r$	Area of cut
$X \cap \partial Y^r$	Boundary length of cut in interior of X
$\partial X \cap Y^r$	Boundary length of cut in boundary of X
$\partial X \cap \partial Y^r$	Point count in boundary intersection

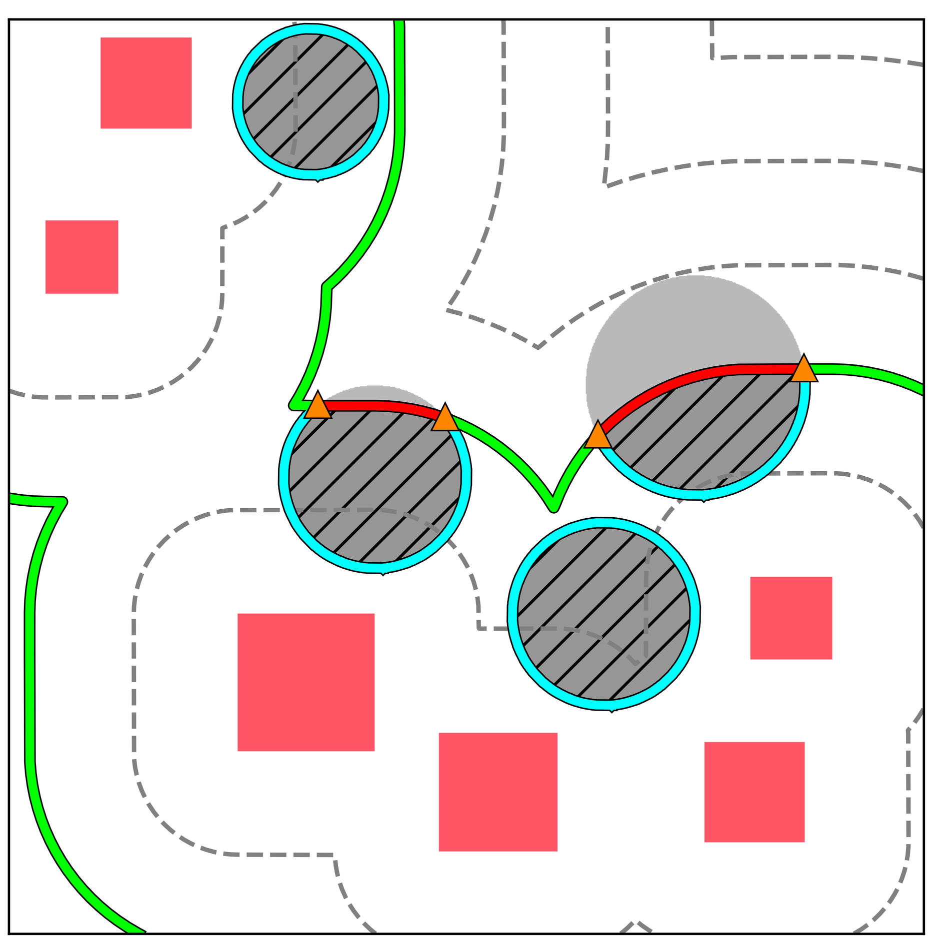


Figure 1: An example where we measure a set of circle objects using the squares as reference. Contour lines indicate the distance from the reference object. For each distance r, we measure the part of the circles within distance r to the squares. The measures are intersection counts (orange triangles), object contour length (blue), intersection contour length (red) and area (hatched region).

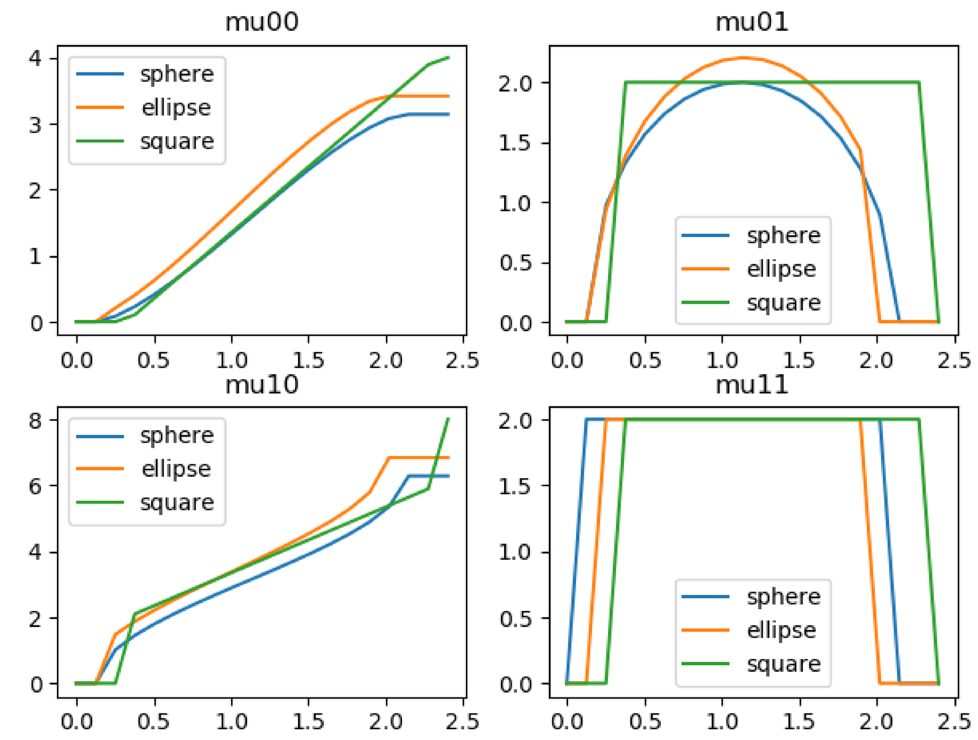


Figure 2: Example measurement curves for simple shapes

A Simpler Case

For this work we will constrain ourselves to the situation where the reference object is the line $x = 0$ and the objects of interest are have shapes which can be described by two functions $p(x)$ and $q(x)$ which together make the contour of the object. Note that if we restrict ourselves to object only on the positive side of $x = 0$, we get $r = x$ as the distance contours. Using the above simplification and letting $p(x) \geq q(x), \ \forall x \in \mathbb{R}^+$, we can reformulate the measures as

$$\mu_{00}(r) = \int_0^r p(x) - q(x) \ dx \tag{3}$$

$$\mu_{01}(r) = p(r) - q(r) \tag{4}$$

$$\mu_{10}(r) = \int_0^r \sqrt{1 + \left(\frac{dp}{dx}\right)^2} + \sqrt{1 + \left(\frac{dq}{dx}\right)^2} dx \tag{5}$$

$$\mu_{11}(r) = \begin{cases} 0 & p(r) = q(r) \\ 2 & \text{otherwise} \end{cases} \tag{6}$$

An Optimization Approach

Letting $\Delta\mu_{mn}(i) = \mu_{mn}(x_{i+1}) - \mu_{mn}(x_i)$ we formulate the following energy terms

$$e_{00} = \sum_{i=1}^{N} \left(\int_{x_i}^{x_{i+1}} p_i - q_i \ dx - \Delta\mu_{00}(x_i) \right)^2 \tag{7}$$

$$e_{01} = \sum_{i=1}^{N} (p_i - q_i - \mu_{01})^2 \tag{8}$$

$$e_{10} = \sum_{i=1}^{N} \left(\int_{x_i}^{x_{i+1}} \sum_{f \in \{p,q\}} \sqrt{1 + \left(\frac{df_i}{dx}\right)^2} - \Delta\mu_{10}(x_i) \right)^2 \tag{9}$$

We further define a smoothness constraint and a curvature minimization terms

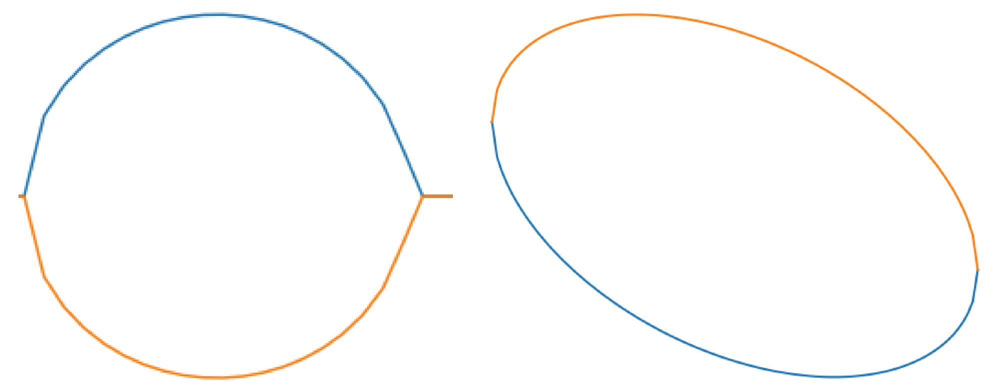
$$e_{\text{smooth}} = \sum_{i=1}^{N-1} \left(\frac{dp_i}{dx}(x_{i+1}) - \frac{dp_{i+1}}{dx}(x_{i+1}) \right)^2 \tag{10}$$

$$+ \sum_{i=1}^{N-1} \left(\frac{dq_i}{dx}(x_{i+1}) - \frac{dq_{i+1}}{dx}(x_{i+1}) \right)^2 \tag{11}$$

$$e_{\text{curve}} = \sum_{f \in \{p,q\}} \left(\int \frac{d^2 f_i}{dx^2} \left(1 + \left(\frac{df_i}{dx}\right)^2 \right)^{-\frac{3}{2}} dx \right)^2 \tag{12}$$

Results

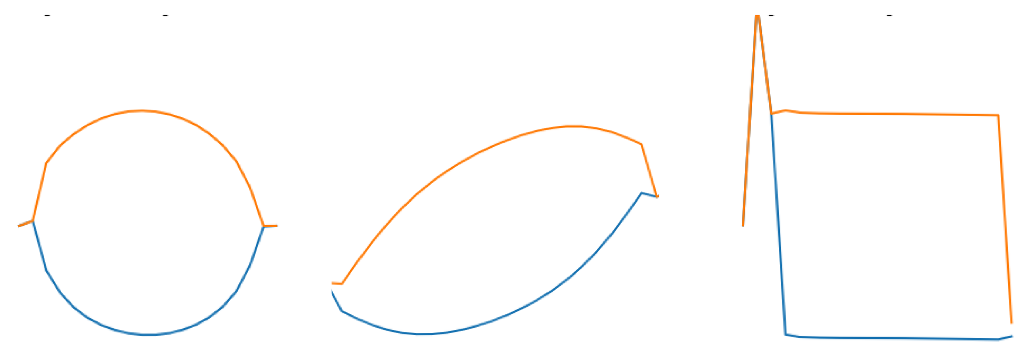
Here we show target example curves corresponding to circle and ellipse. We generated the shape relation measures from these shapes as well as a square.



(a) Circle target **(b)** Ellipse target

Figure 3: Example of target objects which we want to reconstruct.

Using BFGS implemented in scipy.optimize.minimize we minimize the energy terms given the measures. For approximation functions p, q we use 3rd degree splines. The most successful results can be seen below.



(a) Circle reconstruction **(b)** Ellipse reconstruction **(c)** Square reconstruction

Figure 4: Reconstructed objects.

Discussion

While the simplification yields an naively easy way to revert the process to and retrieve the shapes again, we are both far from the goal of inverting the measurements on the full Hausdorff measure in n-d at this point and the optimization approach is currently slow and prone to missing the target, possibly due to local minima in the energy landscape. Furthermore, the following figure shows examples of how unconstrained the problem is yet since both shapes will have equal measure and are thus both solutions to the problem.

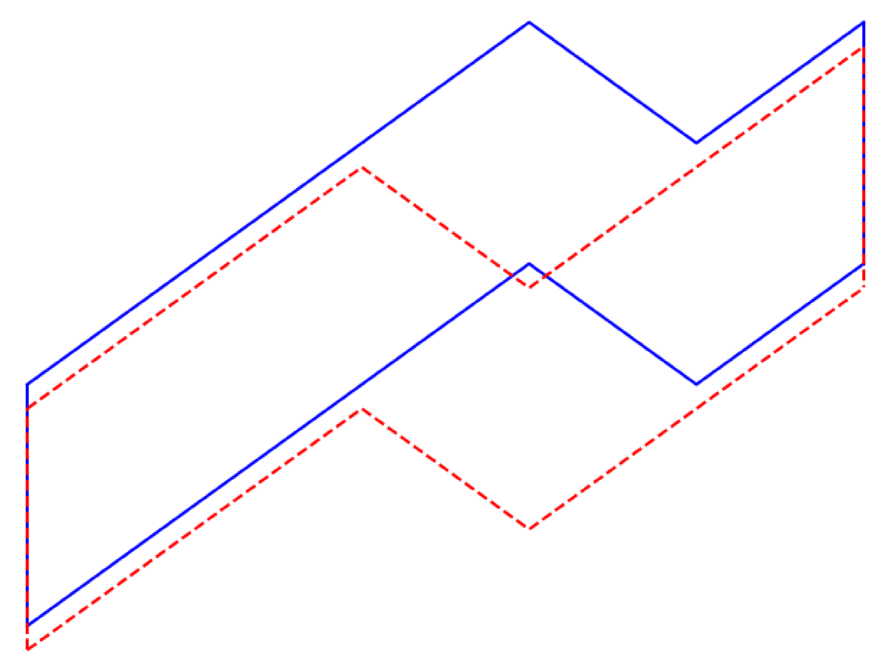


Figure 5: Example of a situation with both desired and less desired equivalence

Identifying these cases yields valuable information about measures themselves.

Open Questions

- Is there a way to analytically retrieve the objects or a parameterized family of objects, that is, the equivalence class under the set of measures?
- What measure could we add that would constrain the object to be have unique shape under a given measure?
- How should we approach the problem if the reference object is not a line, but a point or an object of arbitrary shape?