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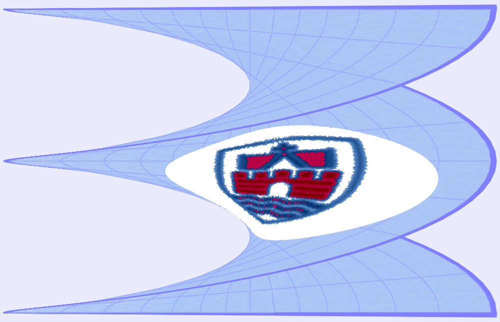
Acousto-electric tomography on Riemannian manifolds

Bjørn Jensen, Kim Knudsen, Steen Markvorsen, and Hjørdis Schlüter

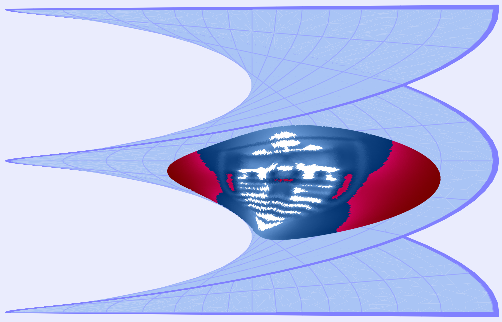
Technical University of Denmark. Contact: hjsc@dtu.dk

Figure 1: Conductivity σ on a helicoid.



Forward map →

Figure 2: One of several power densities $\sigma \cdot |\nabla u|_g^2$ on a helicoid corresponding to different boundary conditions f.



$\phi \uparrow$

$\downarrow \phi^{-1}$

Figure 3: Conductivity σ in the plane.

← Inverse map

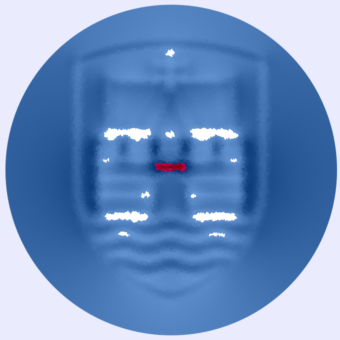


Figure 4: One of several power densities $\left(\frac{\sigma}{\rho^2}\right) \cdot |\nabla u|^2$ in the plane corresponding to different boundary conditions ρf.

1 Introduction

- Acousto-electric tomography (AET) is an imaging modality used to reconstruct the electrical conductivity σ_j^i of an object
- The conductivity enters the model for the electric potential in the object, which on a 2D geometry (M^2, g) is formulated as the BVP:

$$\begin{cases} \Delta_g^\sigma u = 0 & \text{in } \Omega, \\ g(\sigma(\nabla_g u), n_g) = f & \text{on } \partial\Omega \end{cases} \tag{1}$$

  with conductivity σ , potential u, outward unit normal n_g, domain $\Omega \subset M^2$ and using the notation $\Delta_g^\sigma u = \text{div}_g\left(\sigma(\nabla_g(u))\right)$.
- **Inverse problem**: Reconstruct the unknown anisotropic conductivity σ in Ω from several power densities $g(\sigma(\nabla_g u), \nabla_g u)$ corresponding to different boundary conditions f on $\partial\Omega$

2 A step towards the general setting

- We express the PDE in (1) in general coordinates:

$$\begin{aligned} 0 = \Delta_g^\sigma u &= \sum_{mij} \left(\frac{\partial}{\partial x^i}\sigma_j^i\right) \cdot g^{jm} \cdot \frac{\partial u}{\partial x^m} \\ &+ \sum_{mij} \sigma_j^i \cdot \left(\frac{\partial}{\partial x^i} g^{jm}\right) \cdot \frac{\partial u}{\partial x^m} \\ &+ \sum_{mij} \sigma_j^i \cdot g^{jm} \cdot \frac{\partial^2 u}{\partial x^i \partial x^m} \\ &+ \sum_{mij} \Gamma_{ki}^k \cdot \sigma_j^i \cdot g^{jm} \cdot \frac{\partial u}{\partial x^m} \end{aligned}$$

3 A simple illustration

We consider an isotropic conductivity σ on a helicoid (M^2, g), which is conformally represented in the plane with $g = \rho^2 \cdot E$. Specifically, $\Omega \subset M^2$ is parameterized by the map $\phi(x^1, x^2) = (\sinh x^1 \sin x^2, -\sinh x^1 \cos x^2, x^2)$, so that $\rho^2 = \cosh^2(x^1)$

- On a conformal geometry with isotropic conductivity $\sigma = s \cdot E$ the PDE reduces to:

$$\begin{aligned} \Delta_g^\sigma u &= g(\nabla_g u, \nabla_g s) + s \cdot \Delta_g u \\ &= \frac{1}{\rho^2} \cdot \Delta_{g_E}^\sigma u, \end{aligned}$$

  with $g_E = E$. Hence $\Delta_g^\sigma u = 0 \Leftrightarrow \Delta_{g_E}^\sigma u = 0$.
- In the plane, the boundary condition and power densities formulate to $n \cdot \sigma \nabla u = \rho f$ and $\left(\frac{\sigma}{\rho^2}\right) \cdot |\nabla u|^2$
- The reconstruction problem on the helicoid can therefore easily be addressed and handled in its planar, conformal representation

4 Current and future work

- We are currently working on anisotropic conductivities on more general 2D geometries
- Future work: Higher dimensions